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The Effros–Maréchal Topology in the Space of von Neumann Algebras, II

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We continue our study [9] of the Effros–Maréchal topology on the space vN(H) of all von Neumann algebras acting on a fixed separable Hilbert space H. In particular, we prove that factors of each of the types: II_1, II_∞, and (for fixed $\lambda \in [0, 1]$) III_λ, form a dense subset of vN(H). Moreover, the density of type I-factors (or, equivalently, of injective factors) in vN(H), is proved to be equivalent to famous conjectures by A. Connes and E. Kirchberg.

1. INTRODUCTORY PRELIMINARIES

We study the space vN(H) of all von Neumann algebras acting on a fixed separable Hilbert space H, equipped with the weakest topology that makes the function

$$N \mapsto \|\varphi|_N\|$$

continuous on vN(H) for every normal functional φ on $\mathscr{B}(H)$. In the first part of this study [9], we called this topology the *Effros–Maréchal topology* and explained how it arose in papers by E. Effros and O. Maréchal. We further proved several new results about this topology, some of which will be recalled below for the convenience of the reader; a main overall conclusion from these results is that the topology in question seems to be a very natural one for the global study of the theory of von Neumann algebras. It is also quite manageable: convergence in vN(H) is given by computable

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notions of limes inferior and limes superior, it makes vN(H) a Polish space, and it works well with modular theory.

The main theme of this paper is the study of topological properties for the following subsets of vN(H):

- $\mathscr{F}$, the set of factors acting on H;
- $\mathscr{F}_X$, the set of factors of type X (where X is among the standard numberings of the types of factors, such as II_1 or III_λ), acting on H;
- $\mathscr{F}_{\mathrm{inj}}$, the set of injective factors, acting on H;
- $\mathscr{F}^{\mathrm{st}}$, the set of factors acting standardly (cf. [6]) on H;
- The sets $\mathrm{vN}_{\mathrm{fin}}(H)$ and $\mathrm{vN}_{\mathrm{p.i.}}(H)$ of finite (respectively properly infinite) members of vN(H).

Self-explanatory extensions of these notations will also be made, such as $\mathscr{F}^{\mathrm{st}}_{\mathrm{inj}}$ to mean $\mathscr{F}^{\mathrm{st}} \cap \mathscr{F}_{\mathrm{inj}}$, or $\mathrm{vN}^{\mathrm{st}}_{\mathrm{p.i.}}(H)$ to mean properly infinite von Neumann algebras acting standardly on H.

Our main results in this direction (including some results from [9, Section 3–5]) may be summarized as follows, where the symbol * is explained below.

Subset of vN(H)	Dense?	G_δ-set?
$\mathscr{F}$	Yes	Yes
$\bigcup_{n \leqslant n_0} \mathscr{F}_{\mathrm{I}_n}$, $n_0 \in \mathbb{N}$	No	Yes (closed)
$\mathscr{F}_{\mathrm{I}_{\mathrm{fin}}}$	*	No (but F_σ)
$\mathscr{F}_{\mathrm{I}_\infty}$	*	No (but F_σ)
$\mathscr{F}_{\mathrm{II}_1}$	Yes	No
$\mathscr{F}_{\mathrm{II}_\infty}$	Yes	No
$\mathscr{F}_{\mathrm{III}_0}$	Yes	No
$\mathscr{F}_{\mathrm{III}_\lambda}$, $\lambda \in (0, 1)$	Yes	No
$\mathscr{F}_{\mathrm{III}_1}$	Yes	Yes
$\mathscr{F}_{\mathrm{inj}}$	*	Yes
$\mathscr{F}^{\mathrm{st}}$	Yes	Yes

In this table, the properties labeled * are all equivalent, and moreover we prove that they are equivalent to each of the following two statements:

(D) $C^*(\mathbb{F}_\infty)\bigotimes_{\max} C^*(\mathbb{F}_\infty)\cong C^*(\mathbb{F}_\infty)\bigotimes_{\min} C^*(\mathbb{F}_\infty)$

(E) Any separable type II_1-factor can be embedded in R^ω.

Here, $C^*(\mathbb{F}_\infty)$ denotes the full C^*-algebra generated by the free group on infinitely many generators, while R^ω denotes the ultraproduct factor along a free ultrafilter ω on $\mathbb{N}$ associated to the injective type II_1-factor R. (Notice also that, as is costumary, a separable von Neumann algebra means one which has separable predual.) The conjecture (E) arose in Connes' work on the classification of injective factors [3]. The conjecture (D) was studied intensively by Kirchberg [12], who proved its equivalence with (E) and many other conjectures in C^*-algebra theory, all of which remain unsettled. Thus the above points to a somewhat surprising meaning of Effros–Maréchal topology beyond von Neumann algebra theory.

The structure of the paper is as follows: We first deal with density of factors of type III and II in Sections 2 and 3, respectively. The equivalence of density of factors of type I and property (D), is proved in Section 4, and in Section 5, we give a purely von Neumann algebraic proof of the equivalence of density of factors of type I and property (E) (without using property (D) and Kirchberg's result).

In the rest of this section, we recall some crucial notations and results from [9], all of which will be tacitly employed later on. Throughout, H denotes a fixed, separable, infinite dimensional Hilbert-space. The set of bounded operators on H is denoted $\mathscr{B}(H)$, and the set of unitaries in $\mathscr{B}(H)$ is denoted $\mathscr{U}(H)$ (alike notations are used for any Hilbert space). Further, we have the various notations for (sets of) von Neumann algebras introduced above.

We use the following abbreviations for the various topologies on $\mathscr{B}(H)$:

- weak operator topology: wo
- strong operator topology: so
- strong* operator topology: so*

these are also used as subscripts and superscripts in limits, so that, for instance, $x_n \xrightarrow{\text{so}^*} x$ means that (x_n) converges to x in strong* operator topology.

In [9, Section 2], we introduced the notions $\liminf M_\alpha$ and $\limsup M_\alpha$ for a net $(M_\alpha)\subseteq \mathrm{vN}(H)$. In the special case (as assumed here) when H is separable, and (M_n) is a sequence, these definitions may be simplified to:

$$\liminf M_n = \left\{ x\in\mathscr{B}(H) \mid \exists (x_n)\in\prod_n M_n \colon x_n \xrightarrow{\ \text{so}^*\ } x \right\}$$

and

$$\limsup M_n = \left\{ x \in \mathscr{B}(H) \mid \exists (x_n) \in \prod_n M_n \colon \sup_n \|x_n\| < \infty \right.$$
$$\left. \text{and } x \text{ is a wo-limitpoint of } (x_n) \right\}''.$$

(Notice the double commutant in the last equation.) This follows from [9, 2.1, 2.2 and 2.7], using that the so*-topology is second countable on bounded subsets of $\mathscr{B}(H)$.

It is known that $\liminf M_n$ is a von Neumann algebra [9, 2.3], and if convergence of (M_n) to $M \in \mathrm{vN}(H)$ in the Effros–Maréchal topology is denoted as $M_n \to M$, one has:

$$M_n \to M \Leftrightarrow \liminf M_n = M = \limsup M_n.$$

Moreover, one has the commutant theorem [9, 3.5]:

$$\limsup M_n = (\liminf M'_n)',$$

which together with the previous equation reduces convergence questions to computation of $\liminf M_n$ and $\liminf M'_n$.

Finally, we use the following standard notations for the GNS-representation of $M \in \mathrm{vN}(H)$ with respect to a normal faithful state φ on M: $L^2(M, \varphi)$ denotes the completion of M with respect to the inner product $(x, y) \mapsto \varphi(y^*x)$, ξ_φ denotes the copy of the identity operator 1 in $L^2(M, \varphi)$, and $\pi_\varphi \colon M \to \mathscr{B}(L^2(M, \varphi))$ is the representation given by left multiplication on the dense copy of M in $L^2(M, \varphi)$.

2. DENSITY OF TYPE III-FACTORS

We begin by proving the density of (various subclasses of) type III-factors. Here is the strategy for this: we first prove two basic lemmas of "downwards approximation type", stating essentially:

- In an inclusion of standardly acting von Neumann algebras with a normal faithful conditional expectation, the smaller algebra is the limit of algebras spatially isomorphic to the larger one,
- Given any pair of von Neumann algebras, each of the algebras is the limit of algebras spatially isomorphic to the tensor product of the pair.

Using the second lemma, we first prove density of $\mathrm{vN}^{\mathrm{st}}_{\mathrm{p.i.}}(H)$, and then density of $\mathscr{F}$ follows from the first lemma by a certain crossed product trick. Then density of $\mathscr{F}_{\mathrm{III}_1}$ is immediate from the second lemma, and finally density of $\mathscr{F}_{\mathrm{III}_\lambda}$ ($\lambda \neq 1$) follows again by applying the first lemma and a discrete group action.

The following lemma is proved from well-known principles.

2.1. LEMMA. *Let A be a unital C^*-algebra, and let π be a representation of A in $\mathscr{B}(K)$ for some Hilbert space K. Assume (u_n) is a sequence of unitaries on K with $u_n \xrightarrow{\text{so}} v$ and $vv^* \in \pi(A)'$. Then, for every $x \in A$,*

$$u_n^* \pi(x)\, u_n \xrightarrow{\text{so}^*} v^* \pi(x)\, v \qquad \textit{as} \quad n \to \infty.$$

Proof. A straightforward computation proves the stated convergence in weak operator topology. If u is a unitary in A, the condition $vv^* \in \pi(A)'$ ensures that $v^*\pi(u)\, v$ is a unitary, so as weak operator- and strong*-topology coincide on $\mathscr{U}(K)$, we get

$$u_n^* \pi(u)\, u_n \xrightarrow{\text{so}^*} v^* \pi(u)\, v \qquad \text{as} \quad n \to \infty.$$

Since the unitaries span A, the claim follows. ∎

Note. It is a classical exercise to show that the set of isometries coincides with the strong operator closure of the unitary group on a Hilbert space.

2.2. THEOREM. *Let N, $M \in \mathrm{vN}(H)$ be infinite dimensional, with $N \subseteq M$, and assume E is a normal faithful conditional expectation of M onto N. Assume also that we are given a Hilbert space K, and M_1, $N_1 \in \mathrm{vN}(K)$ such that M_1 and N_1 are both standard on K, and moreover $M_1 \cong M$ and $N_1 \cong N$. Then there is a sequence $(u_n) \subseteq \mathscr{U}(K)$ such that $u_n M_1 u_n^* \to N_1$. Moreover, whenever M and N are as stated, we can find K, M_1, N_1 as stated.*

Proof. Let φ be a fixed normal faithful state on N, and define $\psi = \varphi \circ E$. By uniqueness of standard forms [6, 2.3], we may identify M_1 with the GNS-representation of M associated with ψ, meaning that we assume: $K = L^2(M, \psi)$ and $M_1 = \pi_\psi(M)$. Let $K' = L^2(N, \varphi)$, then we may regard K' as a closed subspace of K, and as K and K' are both infinite dimensional, we may take an isometry $v \in \mathscr{B}(K)$ with $v(K) = K'$. Then $v^*\pi_\varphi(N)\, v$ is a standard representation of N on K, so we may assume $N_1 = v^*\pi_\varphi(N)\, v$. By [19, Section 5], the modular involution of J_N of $\pi_\varphi(N)$ on K' is the restriction to K' of the modular involution J_M of $\pi_\psi(M)$ on K, and also

$$\pi_\psi(x)|_{K'} = \pi_\varphi(x), \qquad x \in N.$$

In particular, vv^* commutes with J_M, and $vv^* \in \pi_\psi(N)'$.

Now, let (u_n) be any sequence of unitaries on K satisfying $u_n \xrightarrow{\text{so}} v$. Then, by Lemma 2.1, for all $x \in N$ one has

$$v^*\pi_\varphi(x)\, v = v^*\pi_\psi(x)\, v = \operatorname{so}^* - \lim_{n \to \infty} u_n^* \pi_\psi(x)\, u_n$$

so $N_1 \subseteq \liminf u_n^* M_1 u_n$. Also,

$$\begin{aligned} N_1' &= v^*\pi_\varphi(N)'\, v = v^* J_N \pi_\varphi(N)\, J_N v \\ &= v^* J_M \pi_\psi(N)\, J_M v \end{aligned}$$

and

$$M_1' = J_M \pi_\psi(M)\, J_M.$$

Moreover, as J_M commutes with vv^*, we have $vv^* \in J_M \pi_\psi(N)'\, J_M$, so by the same argument as before, we get $N_1' \subseteq \liminf u_n^* M_1' u_n$. Hence $u_n^* M_1 u_n \to N_1$ by [9, 3.7]. ∎

2.3. Corollary. *Assume that N, $M \in \mathrm{vN}(H)$ are infinite dimensional, that $N \subseteq M$, and that there is a normal faithful conditional expectation of M onto N. Then, if N acts standardly on H, there is $M_0 \in \mathrm{vN}(H)$ and $(v_n) \subseteq \mathscr{U}(H)$ such that $M_0 \cong M$ and $v_n M_0 v_n^* \to N$. If also M acts standardly on H, then we may take $M_0 = M$.*

Proof. Let K, N_1, M_1 and $(u_n) \subseteq \mathscr{U}(K)$ be as in the theorem. By the uniqueness of standard forms, there is a unitary $v\colon H \to K$ such that $vNv^* = N_1$. Let $M_0 = v^* M_1 v$ and $v_n = v^* u_n v$ $(n \in \mathbb{N})$. Then $M_0 \cong M$ and

$$v_n M_0 v_n^* = v^* u_n M_1 u_n^* v \to v^* N_1 v = N.$$

If also M acts standardly on H, then by the proof of Theorem 2.2, we may take $K = H$, $M_1 = M$ and $N_1 = N$. ∎

2.4. Lemma. *Let K be an infinite dimensional separable Hilbert space, and let $v_0\colon H \otimes K \to H$ be a surjective isometry. Then there is a sequence $(u_n) \subseteq \mathscr{U}(H \otimes K)$ such that for any $N \in \mathrm{vN}(H)$ and $M \in \mathrm{vN}(K)$,* one has

$$v_0 u_n^* (N \otimes M)\, u_n v_0^* \to N$$

in $\mathrm{vN}(H)$.

Proof. Fix a unit vector $\xi_0 \in K$ and let $v \in \mathscr{B}(H \otimes K)$ be the isometry given by

$$v\xi = v_0 \xi \otimes \xi_0, \qquad \xi \in H \otimes K.$$

Take a sequence $(u_n) \in \mathscr{U}(H \otimes K)$ with $u_n \xrightarrow{\text{so}} v$. Applying Lemma 2.1 to the representation $x \mapsto x \otimes 1$ of N in $\mathscr{B}(H \otimes K)$, one gets:

$$v_0^* x v_0 = \operatorname*{so^* - lim}_{n \to \infty} u_n^* (x \otimes 1)\, u_n, \qquad x \in N,$$

so

$$v_0^* N v_0 \subseteq \liminf u_n^* (N \otimes M)\, u_n.$$

With $N_1 = v_0^* N v_0 = v^* (N \otimes \mathbb{C}1)\, v \in \mathrm{vN}(H \otimes K)$, we have

$$N_1' = v^* (N' \otimes \mathbb{C}1)\, v,$$

so that the above argument applied to N' and M' gives:

$$N_1' \subseteq \liminf u_n^* (N' \otimes M')\, u_n = \liminf (u_n^* (N \otimes M)\, u_n)'.$$

It follows by [9, Cor. 3.7] that

$$v_0 u_n^* (N \otimes M)\, u_n v_0^* \to N,$$

as desired. ∎

2.5. THEOREM. *The set* $\mathrm{vN}_{\text{p.i.}}^{\text{st}}(H)$ *of properly infinite von Neumann algebras acting standardly on H, is dense in* $\mathrm{vN}(H)$.

Proof. Let $N \in \mathrm{vN}(H)$ be arbitrary. To approximate N by standard properly infinite von Neumann algebras, we apply the trick used in the argument of [9, 6.16]. Namely, let K_1 and K_2 be infinite dimensional separable Hilbert spaces, let $K = K_1 \otimes K_2$, and choose an isometry v_0 of $H \otimes K$ onto H. Take a sequence $(u_n) \in \mathscr{U}(H \otimes K)$ as in Lemma 2.4. With $M = \mathscr{B}(K_1) \otimes \mathbb{C}1_{\mathscr{B}(K_2)}$, we have

$$N \otimes M = N \otimes \mathscr{B}(K_1) \otimes \mathbb{C}1_{\mathscr{B}(K_2)} \in \mathrm{vN}_{\text{p.i.}}(H \otimes K),$$

and

$$(N \otimes M)' = N' \otimes \mathbb{C}1_{\mathscr{B}(K_1)} \otimes \mathscr{B}(K_2) \in \mathrm{vN}_{\text{p.i.}}(H \otimes K),$$

and so $N \otimes M \in \mathrm{vN}_{\text{p.i.}}^{\text{st}}(H \otimes K)$. Hence clearly $v_0 u_n^* (N \otimes M)\, u_n v_0^* \in \mathrm{vN}_{\text{p.i.}}^{\text{st}}(H)$ for every $n \in \mathbb{N}$, and by Lemma 2.4, $v_0 u_n^* (N \otimes M)\, u_n v_0^* \to N$ in $\mathrm{vN}(H)$. ∎

2.6. THEOREM. *The set* $\mathscr{F}^{\text{st}}$ *of factors acting standardly on H, is dense in* $\mathrm{vN}(H)$.

Proof. By the preceding theorem, it suffices to prove that $\mathscr{F}^{\mathrm{st}}$ is dense in $\mathrm{vN}^{\mathrm{st}}_{\mathrm{p.i.}}(H)$. So let $N \in \mathrm{vN}^{\mathrm{st}}_{\mathrm{p.i.}}(H)$ and fix a normal faithful state φ on N. Let

$$\tilde{N} = \bigotimes_{i=1}^{\infty} (N, \varphi),$$

i.e. $\tilde{N}$ is the infinite tensor power of N with respect to φ. The permutation group $\mathbb{S}_\infty$ acts on $\tilde{N}$ by permuting elementary tensors, and this action is ergodic on the center of $\tilde{N}$ (as can be seen from [17, Thm. 5.2.4] in combination with [18, 2.1 and 2.7]). Hence $M = \tilde{N} \rtimes \mathbb{S}_\infty$ is a factor (cf. e.g. [16, 22.6]). There is a normal faithful conditional expectation of $\tilde{N}$ onto

$$N_0 = \{x \otimes 1 \otimes 1 \otimes \cdots \mid c \in N\} \subseteq \tilde{N},$$

and hence there is a normal faithful conditional expectation of M onto the canonical copy of N_0 in M. By Theorem 2.2, there is a Hilbert space K, standardly represented N_1, $M_1 \in \mathrm{vN}(K)$, and a sequence $(u_n) \subseteq \mathscr{U}(K)$, such that $N_1 \cong N$, $M_1 \cong M$ and $u_n M_1 u_n^* \to N_1$ in $\mathrm{vN}(K)$. But as N is standard on H, there is a unitary u: $H \to K$ with $uNu^* = N_1$ in $\mathrm{vN}(K)$, so that $u^* u_n M_1 u_n^* u \to N$ in $\mathrm{vN}(H)$. ∎

To see that $\mathrm{vN}^{\mathrm{st}}(H)$ and $\mathscr{F}^{\mathrm{st}}$ are also G_δ-sets, we need:

2.7. LEMMA. *Let $M \in \mathrm{vN}(H)$. Then M has a cyclic vector if and only if the following condition holds: Given $n \in \mathbb{N}$, $\varepsilon > 0$ and $\eta_1, ..., \eta_n \in H$, there is $\xi \in H$ and $x_1, ..., x_n \in M$ such that*

$$\|\eta_j - x_j \xi\| < \varepsilon, \qquad j = 1, ..., n.$$

Proof. Clearly, the condition is necessary. To see sufficiency, choose a dense sequence (ξ_n) in H. Assuming the condition, we may choose, for each $n \in \mathbb{N}$, a vector $\zeta_n \in H$ and $x_1^{(n)}, ..., x_n^{(n)} \in M$ such that

$$\|\xi_j - x_j^{(n)} \zeta_n\| < \frac{1}{n}, \qquad j = 1, ..., n.$$

For each $n \in \mathbb{N}$ let $p_n \in M'$ be the cyclic projection of M' determined by ζ_n, i.e. $p_n H = \overline{M\zeta_n}$. Then

$$\|(1 - p_n)\, \xi\| = \mathrm{dist}(\xi_j, p_n H) < \frac{1}{n}, \qquad j = 1, ..., n.$$

By density of (ξ_n), it follows that $p_n \xrightarrow{\mathrm{so}} 1$. But as the set of cyclic projections of a von Neumann algebra is so-closed (cf. [11, 7.3.10]), it follows that 1 is a cyclic projection of M', i.e. M has a cyclic vector. ∎

2.8. THEOREM. *The sets* $\mathrm{vN}^{\mathrm{st}}(H)$ *and* $\mathscr{F}^{\mathrm{st}}$ *are* G_δ*-sets in* $\mathrm{vN}(H)$.

Proof. As $\mathscr{F}$ is a G_δ-set [9, 3.11(i)], if suffices to consider $\mathrm{vN}^{\mathrm{st}}(H)$. Let $\mathrm{vN}^{\mathrm{c}}(H)$ denote the set of von Neumann algebras on H which have a cyclic vector, and let $\mathscr{C}$ denote the commutant operation on $\mathrm{vN}(H)$. As H is separable, a standard von Neumann algebra is one which has a cyclic separating vector [6, 2.8], and hence by [20, V.1.14],

$$\mathrm{vN}^{\mathrm{st}}(H)=\mathrm{vN}^{\mathrm{c}}(H)\cap\mathscr{C}(\mathrm{vN}^{\mathrm{c}}(H)).$$

As $\mathscr{C}$ is a homeomorphism of $\mathrm{vN}(H)$ [9, 3.6], it is therefore enough to prove that $\mathrm{vN}^{\mathrm{c}}(H)$ is G_δ.

Let $(\xi_n)\subseteq H$ be a dense sequence, and let (a_n) be as in [9, 3.9], i.e. each a_n is a so*-continuous map from $\mathrm{vN}(H)$ into the unit ball of $\mathscr{B}(H)$; and for each $N\in\mathrm{vN}(H)$, the set $\{a_n(N)\colon n\in\mathbb{N}\}$ is so*-dense in the unit ball of N. With these notations, we now claim:

$$\mathrm{vN}^{\mathrm{c}}(H)=$$

$$\bigcap_{\ell\in\mathbb{N}}\ \bigcap_{n\in\mathbb{N}}\ \bigcap_{\lambda\in\mathbb{N}^n}\ \bigcup_{m\in\mathbb{N}}\ \bigcup_{\mu\in\mathbb{N}^n}\ \bigcup_{t\in[0,\infty)}\ \bigcap_{j=1}^{n}\left\{N\in\mathrm{vN}(H)=\|\xi_{\lambda_j}-ta_{\mu_j}(N)\,\xi_m\|<\frac{1}{\ell}\right\}.$$

Once this is proved, we are done, since the set on the right hand side is clearly G_δ.

To prove the inclusion $\supseteq$ of the above identity, take N belonging to the right hand side; we use Lemma 2.7 to show $N\in\mathrm{vN}^c(H)$. So let $n,\ell\in\mathbb{N}$ and $\eta_1,\ldots,\eta_n\in H$ be given. As (ξ_n) is dense we may take $\lambda\in\mathbb{N}^n$ such that

$$\|\xi_{\lambda_j}-\eta_j\|<\frac{1}{2\ell},\qquad j=1,\ldots,n.$$

Then we choose (using the assumption on N) $m\in\mathbb{N}$, $\mu\in\mathbb{N}^n$ and $t\in[0,\infty)$ such that

$$\|\xi_{\lambda_j}-ta_{\mu_j}(N)\,\xi_m\|<\frac{1}{2\ell},\qquad j=1,\ldots,n.$$

It then follows that, if we put $\xi=\xi_m$ and $x_j=ta_{\mu_j}(N)$ $(j=1,\ldots,n)$, we have

$$\|\eta_j-x_j\xi\|<\frac{1}{\ell},\qquad j=1,\ldots,n,$$

so $N\in\mathrm{vN}^c(H)$.

To see the inclusion $\subseteq$ of the desired identity, let $N \in \mathrm{vN}^c(H)$, and let ℓ, $n \in \mathbb{N}$ and $\lambda \in \mathbb{N}^n$ be given. By Lemma 2.7, we obtain $\xi \in H$ and $x_1, ..., x_n \in N$ such that

$$\|\xi_{\lambda_j} - x_j \xi\| < \frac{1}{3\ell}, \qquad j = 1, ..., n.$$

Let $t = 1 + \max_{j=1, ..., n} \|x_j\|$, and choose $m \in \mathbb{N}$ such that $\|\xi - \xi_m\| < 1/3t\ell$. Next, by so-density of $(a_n(N))$ in the unit ball of N, we choose $\mu \in \mathbb{N}^n$ such that

$$\|a_{\mu_j}(N)\, \xi_m - t^{-1} x_j \xi_m\| < \frac{1}{3t\ell}, \qquad j = 1, ..., n.$$

Then for $j \in \{1, ..., n\}$, we have

$$\begin{aligned} \|\xi_{\lambda_j} - t a_{\mu_j}(N)\, \xi_m\| &\leqslant \|\xi_{\lambda_j} - x_j \xi\| + \|x_j \xi - x_j \xi_m\| + \|x_j \xi_m - t a_{\mu_j}(N)\, \xi_m\| \\ &< \frac{1}{3\ell} + \|x_j\| \frac{1}{3t\ell} + \frac{1}{3\ell} < \frac{1}{\ell}. \end{aligned}$$

Hence N is in the right hand side of the claimed formula for $\mathrm{vN}^c(H)$. ∎

We finally come to the density of type III-factors. We begin with a lemma which, as the proof shows, is implicit in the literature.

2.9. Lemma. *Let $N \in \mathscr{F}_{\mathrm{III}_1}$ and $\lambda \in [0, 1)$. Then there is a discrete subgroup G of $\mathbb{R}$ and an action α of G on N such that $N \rtimes_\alpha G \in \mathscr{F}_{\mathrm{III}_\lambda}$.*

Note. If N is injective, then so is $N \rtimes_\alpha G$ by [3,6.8].

Proof. Fix a normal faithful state φ on N.

If $\lambda = 0$, let $G = \mathbb{Q}$ and let $\alpha_q = \sigma^\varphi_q$, $q \in \mathbb{Q}$. As $T(N) = \{0\}$ by [2, 3.4.1] (this reference also explains the invariant T), the action α is outer on N, so $M = N \rtimes_\alpha \mathbb{Q}$ is a factor, and by the proof of [2, 1.5.8(a)] one has $T(M) = \mathbb{Q}$. Hence by [2, 3.4.1] again, M is of type III_0.

If $\lambda \in (0, 1)$, let $t_0 = -2\pi/\log \lambda$ and $\alpha = \sigma^\varphi_{t_0}$. Then as before, α defines an outer action of $\mathbb{Z}$ on N, and with $M = N \rtimes_\alpha \mathbb{Z}$, one has $T(M) = t_0 \mathbb{Z}$; so by [2, 3.4.1], $M \in \mathscr{F}_{\mathrm{III}_0} \cup \mathscr{F}_{\mathrm{III}_\lambda}$. Let E: $M \to N$ be the canonical conditional expectation, and put $\psi = \varphi \circ E$. Then, by [14, 4.2], [15, 2.1], one has a very explicit spatial isomorphism

$$M \rtimes_{\sigma^\psi} \mathbb{R} \cong (N \rtimes_{\sigma^\varphi} \mathbb{R}) \rtimes_{\bar{\alpha}} \mathbb{Z},$$

where $\bar{\alpha}$ is the obvious (inner) extension of α to the factor $N \rtimes_{\sigma^\varphi} \mathbb{R}$. Hence

$$\mathscr{Z}(M \rtimes_{\sigma^\psi} \mathbb{R}) \cong \mathbb{C}1 \otimes \mathscr{L}(\mathbb{Z}) \cong L^\infty(\mathbb{R}/\mathbb{Z}),$$

and under the above spatial isomorphism, it is easy to see that the dual action of σ^ψ restricted to this center is just the translation action of $\mathbb{R}$ on $L^\infty(\mathbb{R}/\mathbb{Z})$. Hence the flow of weights of M is transitive and $M \in \mathscr{F}_{\mathrm{III}_\lambda}$. ∎

2.10. THEOREM. *For every* $\lambda \in [0, 1]$, *one has*:

(i) $\mathscr{F}_{\mathrm{III}_\lambda}$ *is dense in* $\mathrm{vN}(H)$;

(ii) $\mathscr{F}_{\mathrm{III}_\lambda} \cap \mathscr{F}_{\mathrm{inj}}$ *is dense in* $\mathscr{F}_{\mathrm{inj}}$.

In particular, $\mathscr{F}_{\mathrm{III}_\lambda}$ *is not* G_δ *in* $\mathscr{F}$ *if* $\lambda \neq 1$.

Proof. We first consider (i) in the case $\lambda = 1$. By Theorem 2.6, it suffices to prove that $\mathscr{F}_{\mathrm{III}_1}$ is dense in $\mathscr{F}$. So let $N \in \mathscr{F}$, and let the injective type III_1-factor R_∞ act on a separable Hilbert space K. Let $v_0 \colon H \otimes K \to H$ be a surjective isometry. By Lemma 2.4, there is a sequence $(u_n) \subseteq \mathscr{U}(H \otimes K)$ such that $v_0 u_n^*(N \otimes R_\infty)\, u_n v_0^* \to N$, and clearly $v_0 u_n^*(N \otimes R_\infty)\, u_n v_0^*$ is a type III_1-factor for each $n \in \mathbb{N}$; furthermore, injectivity of N implies injectivity of each of these factors. So the theorem holds for $\lambda = 1$.

For $\lambda \in [0, 1)$ it then suffices (by Theorem 2.6) to prove that

$$\mathscr{F}_{\mathrm{III}_1} \subseteq \overline{\mathscr{F}_{\mathrm{III}_\lambda}} \qquad \text{and} \qquad \mathscr{F}_{\mathrm{III}_1} \cap \mathscr{F}_{\mathrm{inj}} \subseteq \overline{\mathscr{F}_{\mathrm{III}_\lambda} \cap \mathscr{F}_{\mathrm{inj}}},$$

and this follows from Corollary 2.3 and Lemma 2.9.

The last assertion follows from Baire's theorem and the fact [9, 4.9] that $\mathscr{F}_{\mathrm{III}_1}$ is G_δ in $\mathscr{F}$, since $\mathscr{F}_{\mathrm{III}_1} \cap \mathscr{F}_{\mathrm{III}_\lambda} = \varnothing$ $(\lambda \neq 1)$. ∎

2.11. COROLLARY. *None of the sets* $\mathscr{F}_{\mathrm{I}_{\mathrm{fin}}}$, $\mathscr{F}_{\mathrm{I}_\infty}$ *and* $\mathscr{F}_{\mathrm{I}}$ *are* G_δ.

Proof. By [9, 4.9 and 5.2] and (ii) above, $\mathscr{F}_{\mathrm{III}_1} \cap \mathscr{F}_{\mathrm{inj}}$ is a dense G_δ-set in $\mathscr{F}_{\mathrm{inj}}$, and it is clearly disjoint from each of the three sets in the corollary. So by Baire's theorem, it suffices to show that each of those three sets are dense in $\mathscr{F}_{\mathrm{inj}}$.

By Connes' celebrated theorem [3], injective factors are generated by increasing sequences of finite type I-factors, so by [9, 2.8] it follows that $\mathscr{F}_{\mathrm{I}_{\mathrm{fin}}}$ (and hence $\mathscr{F}_{\mathrm{I}}$) is dense in $\mathscr{F}_{\mathrm{inj}}$. Then using Lemma 2.4 with $M = \mathscr{B}(K)$, one gets easily that $\mathscr{F}_{\mathrm{I}_{\mathrm{fin}}}$ and hence $\mathscr{F}_{\mathrm{inj}}$ is contained in the closure of $\mathscr{F}_{\mathrm{I}_\infty}$. ∎

In the light of Theorem 2.10(ii), the question of whether $\mathscr{F}_{\mathrm{inj}}$ is dense in $\mathscr{F}$ seems quite interesting; an affirmative answer would imply, by the above, that the orbit (with respect to isomorphism) of any hyperfinite type

III_λ-factor ($\lambda \neq 0$) is dense in $\mathrm{vN}(H)$. This question will be discussed in Section 4 and 5.

3. DENSITY OF TYPE II-FACTORS

It is known ([9, 4.5]) that $\mathscr{F}_{\mathrm{II}}$ is an F_σ-subset of $\mathscr{F}$, and that each of $\mathscr{F}_{\mathrm{II}_1}$ and $\mathscr{F}_{\mathrm{II}_\infty}$ are differences of G_δ-sets. In this section, we prove that none of $\mathscr{F}_{\mathrm{II}}$, $\mathscr{F}_{\mathrm{II}_1}$ and $\mathscr{F}_{\mathrm{II}_\infty}$ are G_δ-sets, and in fact each of them are dense in $\mathrm{vN}(H)$.

The strategy to prove density of $\mathscr{F}_{\mathrm{II}_1}$ is to show the inclusions

$$\mathscr{F}_{\mathrm{III}_0} \subseteq \overline{\mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_\infty}(H)} \subseteq \overline{\mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_1}(H)} \subseteq \overline{\mathscr{F}_{\mathrm{II}_1}},$$

and appeal to the density of $\mathscr{F}_{\mathrm{III}_0}$ proved in Section 2. Then density of $\mathscr{F}_{\mathrm{II}_\infty}$ follows easily using Lemma 2.4 and the density of $\mathscr{F}_{\mathrm{III}_1}$.

We shall use repeatedly the following fact, which follows from [9, 2.8]: if $(N_n) \subseteq \mathrm{vN}(H)$ is an increasing sequence, then the sequence converges in Effros–Maréchal topology to $\bigvee_n N_n$, the von Neumann algebra generated by $\bigcup_n N_n$.

3.1. LEMMA. *The set* $\mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_1}(H)$ *is in the closure of* $\mathscr{F}^{\mathrm{st}}_{\mathrm{II}_1}$.

Proof. Let $N \in \mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_1}(H)$ and let φ be a normal faithful tracial state on N. Using the construction from the proof of Theorem 2.6, the infinite tensor power $\tilde{N}$ of N with respect to φ is a von Neumann algebra of type II_1, and $M = \tilde{N} \rtimes \mathbb{S}_\infty$ is a type II_1-factor. The same argument as in the proof mentioned then shows that N is a limit of standard type II_1-factors (isomorphic to M). ∎

3.2. LEMMA. *Assume* $(q_n) \subseteq \mathscr{B}(H)$ *is an ascending sequence of infinite dimensional projections with* $q \xrightarrow{\mathrm{so}} 1$. *Then there exists a sequence* $(v_n) \subseteq \mathscr{B}(H)$ *of isometries with* $v_n v_n^* = q_n$ $(n \in \mathbb{N})$, *and* $v_n \xrightarrow{\mathrm{so}} 1$.

Proof. Put $q_0 = 0$. For each $k \in \mathbb{N}$, we may choose an ascending sequence $(s_{k,n})_{n \in \mathbb{N}}$ of finite dimensional projections satisfying $s_{k,n} \to q_k - q_{k-1}$ as $n \to \infty$, and $s_{k,n} \leqslant q_k$ for all $k, n \in \mathbb{N}$. Then $r_n = \sum_{k=1}^n s_{k,n}$ is a finite dimensional projection for each $n \in \mathbb{N}$, and

$$r_n \to \sum_{k=1}^{\infty} (q_k - q_{k-1}) = \lim_{k \to \infty} q_k = 1.$$

For $n \in \mathbb{N}$, $\dim(q_n - r_n) = \infty$, hence we may choose an isometry v_n of H onto $q_n H$ which leaves $r_n H$ pointwise fixed; the last condition implies, as $\lim_{n \to \infty} r_n = 1$, that $v_n \xrightarrow{\mathrm{so}} 1$. ∎

3.3. Lemma. *The set* $\mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_\infty}(H)$ *is in the closure of* $\mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_1}(H)$.

Proof. Let $M \in \mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_\infty}(H)$, and let (p_n) be an increasing sequence of infinite dimensional finite projections in M satisfying $p_n \xrightarrow{\mathrm{so}} 1$. Let J be the modular involution on H associated to M, and let

$$q_n = p_n J p_n J, \qquad n \in \mathbb{N}.$$

According to [6, 2.5, 2.6], one has that $q_n M q_n$ is a standard von Neumann algebra on $q_n H$, and $q_n M q_n \cong p_n M p_n \in \mathrm{vN}_{\mathrm{II}_1}(H)$. In particular each q_n is infinite dimensional, and $q_n \xrightarrow{\mathrm{so}} 1$; also, it is easy to check that (q_n) is an increasing sequence of projections. By Lemma 3.2, we may take a sequence (v_n) of isometries in $\mathscr{B}(H)$ such that $v_n v_n^* = q_n$, $(n \in \mathbb{N})$ and $v_n \xrightarrow{\mathrm{so}} 1$. Then $v_n^* x v_n \xrightarrow{\mathrm{so}^*} x$ for every $x \in \mathscr{B}(H)$, in fact the strong convergence follows from

$$\begin{aligned} \|v_n^* x v_n \xi - x\xi\| &= \|v_n^* x v_n \xi - v_n^* v_n x \xi\| \\ &\leqslant \|x(v_n \xi - \xi)\| + \|x\xi - v_n x \xi\|, \qquad \xi \in H, \end{aligned}$$

and then strong*-convergence follows immediately.

Thus with $M_n = v_n^* q_n M q_n v_n \in \mathrm{vN}(H)$ $(n \in \mathbb{N})$, one has $M \subseteq \liminf M_n$, and as $M'_n = v_n^* q_n M' q_n v_n$ for all $n \in \mathbb{N}$, the same argument gives $M' \subseteq \liminf M'_n$, so that $M_n \to M$. As $q_n M q_n \in \mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_1}(H)$, it is clear that $M_n \in \mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_1}(H)$. ∎

3.4. Lemma. *The set* $\mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_\infty}(H)$ *is dense in* $\mathrm{vN}(H)$.

Proof. By Theorem 2.10(i), it suffices to prove that the closure of $\mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_\infty}(H)$ contains $\mathscr{F}_{\mathrm{III}_0}$. With $M \in \mathscr{F}_{\mathrm{III}_0}$, one has by [8, 8.3] (cf. also [2, 5.3.6]) that M is generated by an increasing sequence of type II_∞-subalgebras, say (M_n); in particular, $M_n \to M$. Letting K and L be separable, infinite dimensional Hilbert-spaces, one has

$$M_n \otimes \mathscr{B}(K) \otimes \mathbb{C}1 \in \mathrm{vN}^{\mathrm{st}}_{\mathrm{II}_\infty}(H \otimes K \otimes L),$$

and, using [9, 3.4]

$$M_n \otimes \mathscr{B}(K) \otimes \mathbb{C}1 \to M \otimes \mathscr{B}(K) \otimes \mathbb{C}1$$

in $\mathrm{vN}(H \otimes K \otimes L)$. As M is of type III, one has $M \cong M \otimes \mathscr{B}(K) \otimes \mathbb{C}1$ spatially, and using this spatial isomorphism, it follows that M is the limit of *standard* von Neumann algebras of type II_∞. Q.E.D

3.5. THEOREM. *Both* $\mathscr{F}^{\mathrm{st}}_{\mathrm{II}_1}$ *and* $\mathscr{F}^{\mathrm{st}}_{\mathrm{II}_\infty}$ *are dense in* $\mathrm{vN}(H)$.

Proof. It follows immediately from Lemmas 3.1, 3.3 and 3.4 that $\mathscr{F}^{\mathrm{st}}_{\mathrm{II}_1}$ is dense. To prove density of $\mathscr{F}^{\mathrm{st}}_{\mathrm{II}_\infty}$, it suffices to see that its closure contains $\mathscr{F}_{\mathrm{III}_1}$ (by Theorem 2.10). So let $M \in \mathscr{F}_{\mathrm{III}_1}$ and choose $(A_k) \subseteq \mathscr{F}_{\mathrm{II}_1}$ with $A_k \to M$. Letting K, L be separable, infinite dimensional Hilbert-spaces, one has $A_k \otimes \mathscr{B}(K) \otimes \mathbb{C}1_{\mathscr{B}(L)} \to M \otimes \mathscr{B}(K) \otimes \mathbb{C}1_{\mathscr{B}(L)} \cong M$; since the last isomorphism is spatial, and $A_k \otimes \mathscr{B}(K) \otimes \mathbb{C}1_{\mathscr{B}(L)}$ is standard on $H \otimes K \otimes L$, it follows that M is itself a limit of standard type II_∞-factors. ∎

3.6. COROLLARY. *None of* $\mathscr{F}_{\mathrm{II}_1}$ *and* $\mathscr{F}_{\mathrm{II}_\infty}$ *are* G_δ *in* $\mathrm{vN}(H)$, *but both are dense.*

Proof. Follows from Theorem 3.5, Theorem 2.10 and Baire's theorem. Q.E.D

4. ON DENSITY OF TYPE I-FACTORS

In [9, Section 5] we proved that the set $\mathscr{F}_{\mathrm{inj}}$ of injective factors acting on H, is a non-closed G_δ-subset of $\mathscr{F}$. The question, "how large is $\mathscr{F}_{\mathrm{inj}}$ within $\mathscr{F}$," was left open. This question is related to some famous conjectures in operator algebra theory, as we shall prove in this section.

The main theorem here is as follows; as usual, $C^*(\mathbb{F}_\infty)$ denotes the full C^*-algebra generated by the free group $\mathbb{F}_\infty$.

4.1. THEOREM. *The following statements are equivalent*:

(A) $\mathscr{F}_{\mathrm{I}_{\mathrm{fin}}}$ *is dense in* $\mathrm{vN}(H)$.

(B) $\mathscr{F}_{\mathrm{I}}$ *is dense in* $\mathrm{vN}(H)$.

(C) $\mathscr{F}_{\mathrm{inj}}$ *is dense in* $\mathrm{vN}(H)$.

(D) $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) \cong C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$.

4.2. *Remarks.* The property (D) is, by the work of Kirchberg [12, Section 8], known to be equivalent to a number of other properties, including the following conjecture by Connes [3, p. 105]:

(E) Any separable type II_1-factor can be embedded in R^ω,

where R^ω denotes the ultraproduct von Neumann algebra associated with the hyperfinite II_1-factor along some fixed free ultrafilter ω on $\mathbb{N}$. None of the properties (A)–(E) are presently known to be true or false.

The last condition (E) is also related to Voiculescu's definition of entropy, as mentioned in [22, 7.4].

Notice that the properties (C) and (E) are purely von Neumann algebraic, and their equivalence as claimed above follows only via property

(D), hence using Kirchberg's deep theorem. As an application of more general ultraproduct techniques, we give, in Section 5, a purely von Neumann algebraic proof of the equivalence of (C) and (E).

Proof of Theorem 4.1. As $\mathcal{F}_{\mathrm{I_{fin}}} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{\mathrm{inj}}$, the implications (A) $\Rightarrow$ (B) $\Rightarrow$ (C) are trivial.

(C) $\Rightarrow$ (A): As argued in the proof of Corollary 2.11, it follows from [3] that $\mathcal{F}_{\mathrm{I_{fin}}}$ is dense in $\mathcal{F}_{\mathrm{inj}}$.

(A) $\Rightarrow$ (D): Let $C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty)$ denote the algebraic tensor product, and let π be any representation of this *-algebra in $\mathcal{B}(H)$. Put

$$A = \pi(C^*(\mathbb{F}_\infty) \odot \mathbb{C}1), \qquad B = \pi(\mathbb{C}1 \odot C^*(\mathbb{F}_\infty)).$$

Let (u_n) be a norm-dense sequence of unitaries in the unit ball of $C^*(\mathbb{F}_\infty)$, and let

$$v_i = \pi(u_i \otimes 1) \in A, \qquad w_i = \pi(1 \otimes u_i) \in B.$$

By (A), $M = A''$ is in the closure of $\mathcal{F}_{\mathrm{I_{fin}}}$, so take a sequence $(F_n) \subseteq \mathcal{F}_{\mathrm{I_{fin}}}$ satisfying $F_n \to M$. In particular

$$A \subseteq \liminf F_n$$

and

$$B \subseteq A' = M' = \liminf F'_n,$$

so we may, for all $i, n \in \mathbb{N}$, take unitaries $v_{i,n} \in F_n$ and $w_{i,n} \in F'_n$ such that

$$v_{i,n} \xrightarrow{\text{so*}} v_i \qquad \text{and} \qquad w_{i,n} \xrightarrow{\text{so*}} w_i \qquad \text{for} \quad n \to \infty$$

for all $i \in \mathbb{N}$. By the universal property [20, IV.4.7] of the maximal tensor product, there are unique representations π_n of $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$ in $\mathcal{B}(H)$ with

$$\pi_n(u_i \otimes 1) = v_{i,n}, \qquad i, n \in \mathbb{N}$$

and

$$\pi_n(1 \otimes u_i) = w_{i,n}, \qquad i, n \in \mathbb{N}.$$

As each $(F_n) \subseteq \mathcal{F}_{\mathrm{I_{fin}}}$ one has

$$C^*(F_n, F'_n) \cong F_n \otimes_{\min} F'_n, \qquad n \in \mathbb{N}.$$

Upon identifying these, it is clear that $\pi_n = \sigma_n \otimes \varrho_n$ for each n, where σ_n and ϱ_n are representations of $C^*(\mathbb{F}_\infty)$ in $C^*(F_n, F'_n)$. Hence, for all $n \in \mathbb{N}$,

$$\|\pi_n(x)\| \leqslant \|x\|_{\min} = \sup_{\varrho, \sigma} \|(\varrho \otimes \sigma)(x)\|, \qquad x \in C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty),$$

where the supremum is taken over all representations ϱ, σ of $C^*(\mathbb{F}_\infty)$ in $\mathscr{B}(H)$. On the other hand, by construction, the sequence (π_n) converges to π in the (strong*operator) pointwise sense. Therefore, for each $x \in C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty)$,

$$\|\pi(x)\| \leqslant \liminf_{n \to \infty} \|\pi_n(x)\| \leqslant \|x\|_{\min}.$$

As π was an arbitrary representation, we conclude

$$\|x\|_{\max} \leqslant \|x\|_{\min}, \qquad x \in C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty),$$

hence (D).

Finally, to prove that (D) implies (C), we need the following consequence of Voiculescu's non-commutative Weyl–von Neumann theorem. In the following, the symbol $\sim$ indicates unitary equivalence of representations.

4.3. Lemma. *Let A be a unital C^*-algebra, and let λ and ϱ be representations of A in $\mathscr{B}(H)$. Assume ϱ is faithful and satisfies*

$$\varrho \sim \varrho \oplus \varrho \oplus \cdots.$$

Then there is a sequence $(u_n) \subseteq \mathscr{U}(H)$ such that

$$u_n \varrho(x) u_n^* \xrightarrow{\text{so}^*} \lambda(x), \qquad x \in A.$$

Proof. Let $H_2 = H \oplus H$, $H_\infty = H \oplus H \oplus \cdots$, and $\varrho_\infty = \varrho \oplus \varrho \oplus \cdots$ (acting on H_∞). Clearly $\varrho \oplus \varrho$ and $\lambda \oplus \varrho$ are faithful representations of A on H_2. Now let π, π_2 be the projections of $\mathscr{B}(H)$ (respectively $\mathscr{B}(H_2)$) onto the corresponding Calkin algebras $\mathscr{B}(H)/\mathscr{K}(H)$ and $\mathscr{B}(H_2)/\mathscr{K}(H_2)$, respectively. Then $\pi \circ \varrho$ is faithful because $\varrho \sim \varrho_\infty$ and, for $x \in A$, $\varrho_\infty(x)$ is compact only if $\varrho(x) = 0$. Hence $\pi_2 \circ (\varrho \oplus \varrho) = \pi \circ \varrho \oplus \pi \circ \varrho$ and $\pi_2 \circ (\lambda \oplus \varrho) = \pi \circ \lambda \oplus \pi \circ \varrho$ are also faithful. It now follows from [21, 1.4] that $\varrho \oplus \varrho$ and $\lambda \oplus \varrho$ are approximately unitarily equivalent, i.e. that we have a sequence $(v_n) \subseteq \mathscr{U}(H_2)$ such that

$$\|v_n(\varrho \oplus \varrho)(x) - (\lambda \oplus \varrho)(x) v_n\| \to 0, \qquad x \in A.$$

Next, let $w \in \mathscr{B}(H_2)$ be an isometry with range $H \oplus \{0\}$, and take $(w_n) \subseteq \mathscr{U}(H_2)$ with $w_n \xrightarrow{\text{so}} w$. With λ' defined by

$$\lambda'(x) = w^*(\lambda(x) \oplus 0)\, w, \qquad x \in A,$$

we have $\lambda' \sim \lambda$. As also $\varrho \sim \varrho_\infty \sim \varrho \oplus \varrho$, it suffices to prove the conclusion of the theorem for λ' and $\varrho \oplus \varrho$ in the place of λ and ϱ. But for $x \in A$, we have

$$\begin{aligned} \lambda'(x) &= w^*(\lambda \oplus 0)(x)\, w = w^*(\lambda \oplus \varrho)(x)\, w \\ &= \operatorname{wo-}\lim_{n \to \infty} w_n^* v_n (\lambda \oplus \varrho)(x)\, v_n^* w_n \\ &= \operatorname{wo-}\lim_{n \to \infty} w_n^* v_n (\varrho \oplus \varrho)(x)\, v_n^* w_n. \end{aligned}$$

Then, with $u_n = w_n^* v_n$ $(n \in \mathbb{N})$, we may now invoke Lemma 2.1 to conclude that

$$\lambda'(x) = \operatorname{so^*-}\lim_{n \to \infty} u_n (\varrho \oplus \varrho)(x)\, u_n^*, \qquad x \in A.$$

Q.E.D

Proof that (D) *implies* (C) *in Theorem* 4.1. By a theorem of Choi [1, Theorem 7], there is a sequence (σ_n) of finite dimensional representations of $C^*(\mathbb{F}_\infty)$ such that $\sigma = \sigma \oplus \sigma_2 \oplus \cdots$ is faithful. Replacing σ by

$$\bigoplus_{n=1}^{\infty} \bigoplus_{k=1}^{\infty} \sigma_{n,k} \qquad \text{where} \quad \sigma_{n,k} = \sigma_n \quad (n, k \in \mathbb{N}),$$

we see that we may assume that $\sigma \sim \sigma \oplus \sigma \oplus \cdots$. Thus $\varrho = \sigma \otimes \sigma$ is a faithful (cf. [20, IV.4.9]) representation of $C^*(\mathbb{F}_\infty) \bigotimes_{\min} C^*(\mathbb{F}_\infty)$, it is a direct sum of finite dimensional representations, and satisfies $\varrho \sim \varrho \oplus \varrho \oplus \cdots$. As ϱ is clearly separable, we may further decide that ϱ has image in $\mathscr{B}(H)$.

Now, given $M \in \mathrm{vN}(H)$, choose so*-dense sequences (v_n) of unitaries in M and (w_n) of unitaries in M'. Also let (z_n) be the universal unitaries representing $\mathbb{F}_\infty$ in $C^*(\mathbb{F}_\infty)$. Then, by the assumption that (D) holds, we get from [20, IV.4.7] a unique representation λ of $C^*(\mathbb{F}_\infty) \bigotimes_{\min} C^*(\mathbb{F}_\infty)$ in $\mathscr{B}(H)$ satisfying

$$\lambda(z_n \otimes 1) = v_n \qquad \text{and} \qquad \lambda(1 \otimes z_n) = w_n \qquad \text{for all} \quad n \in \mathbb{N}.$$

By Lemma 4.3, we have $(u_n) \subseteq \mathscr{U}(H)$ with

$$u_n \varrho(x)\, u_n^* \xrightarrow{\text{so}} \lambda(x), \qquad x \in C^*(\mathbb{F}_\infty) \bigotimes_{\min} C^*(\mathbb{F}_\infty).$$

If we define

$$M_n = u_n \varrho(C^*(\mathbb{F}_\infty) \otimes \mathbb{C}1)''\, u_n^*, \qquad n \in \mathbb{N},$$

then

$$M_n' \supseteq u_n \varrho(\mathbb{C}1 \otimes C^*(\mathbb{F}_\infty))\, u_n^*, \qquad n \in \mathbb{N},$$

so we have

$$\lambda(C^*(\mathbb{F}_\infty) \otimes \mathbb{C}1) \subseteq \liminf M_n$$

and

$$\lambda(\mathbb{C}1 \otimes C^*(\mathbb{F}_\infty)) \subseteq \liminf M_n'.$$

Hence (cf. [9, 2.3])

$$M = \lambda(C^*(\mathbb{F}_\infty) \otimes \mathbb{C}1)'' \subseteq \liminf M_n$$

and

$$M' = \lambda(\mathbb{C}1 \otimes C^*(\mathbb{F}_\infty))'' \subseteq \liminf M_n',$$

so $M_n \to M$. Also each M_n is, by the choice of ϱ, a von Neumann algebra of type I, so $\mathrm{vN}_{\mathrm{I}}(H)$ is dense in $\mathrm{vN}(H)$. In particular $\mathrm{vN}_{\mathrm{inj}}(H)$ is dense in $\mathrm{vN}(H)$. By [9, 5.2], $\mathrm{vN}_{\mathrm{inj}}(H)$ is a G_δ-subset of $\mathrm{vN}(H)$, and by [9, 3.11] and Theorem 2.5 here, $\mathscr{F}$ is a dense G_δ-subset of $\mathrm{vN}(H)$. Hence we may use Baire's theorem to conclude (C). ∎

5. EMBEDDING IN THE ULTRAPRODUCT FACTOR AND DENSITY

We now give a direct, von Neumann algebraic proof of the result mentioned (and, upon appealing to Kirchberg's result, proved) in the previous section, namely that density of the set of injective factors in $\mathrm{vN}(H)$ is equivalent to the possibility of embedding any II_1-factor in R^ω, the ultraproduct factor associated with the hyperfinite II_1-factor R. Of course, together with the proof of Theorem 4.1, this may also be regarded as an alternative proof of Kirchberg's result. In fact, our main result here (Theorem 5.8) says that a II_1-factor N is in the closure of $\mathscr{F}_{\mathrm{inj}}$ precisely when N embeds in R^ω, which would be of interest also if the conjectures (A)–(E) of Section 4 should turn out to be false.

Throughout this section, ω denotes a fixed free ultrafilter on $\mathbb{N}$.

We first consider a sequence (H_n) of separable Hilbert spaces. Then $(H_n)_\omega$ is the quotient space of bounded sequences $(\xi_n) \in \prod_{n \in \mathbb{N}} H_n$ by the closed ideal of sequences that converge to zero along ω. For $\xi \in (H_n)_\omega$, we write $\xi = (\xi_n)_\omega$ if $(\xi_n) \in \prod_{n \in \mathbb{N}} H_n$ is a representing sequence for ξ. It is easy to see (cf. e.g. [10, Section 2]) that $(H_n)_\omega$ is a Hilbert space with inner product given by

$$\langle \xi, \eta \rangle = \lim_{n \to \omega} \langle \xi_n, \eta_n \rangle \quad \text{when} \quad \xi, \eta \in (H_n)_\omega \quad \text{and} \quad \xi = (\xi_n)_\omega, \quad \eta = (\eta_n)_\omega$$

(in particular, the limit exists as the sequences are bounded). In fact, the corresponding norm on $(H_n)_\omega$ is nothing but the Banach space quotient norm.

We prove that the quotient map $(\xi_n) \mapsto (\xi_n)_\omega$ has a right inverse on every separable subspace of $(H_n)_\omega$.

5.1. LEMMA. *Let $(H_n)_\omega$ be as above, and let K be an infinite dimensional separable Hilbert-subspace of $(H_n)_\omega$. Then there are unitaries $u_n \in \mathcal{U}(K, H_n)$ for each $n \in \mathbb{N}$ such that, for every $\xi \in K$, one has $\xi = (u_n \xi)_\omega$.*

Proof. Let $(\xi_k)_{k \in \mathbb{N}}$ be an orthonormal basis for K, with $\xi_k = (\xi_k^n)_\omega$. This means that for all $i, j \in \mathbb{N}$,

$$\delta_{i,j} = \langle \xi_i, \xi_j \rangle = \lim_{n \to \omega} \langle \xi_i^n, \xi_j^n \rangle,$$

where $\delta_{i,j}$ is Kronecker's delta. Define for each $k \in \mathbb{N}$:

$$F_k = \{ n \in \mathbb{N} \mid \xi_1^n, ..., \xi_k^n \text{ are linearly independent} \}.$$

Then by the above, each F_k belongs to ω, and $F_1 \supseteq F_2 \supseteq \cdots$. Defining further $G_1 = \mathbb{N}$ and

$$G_k = F_k \cap (\mathbb{N} \setminus \{1, ..., k\}), \qquad k = 2, 3, ...$$

one still has $G_k \in \omega$ for all k (because ω is a free ultrafilter), $G_1 \supseteq G_2 \supseteq \cdots$, and also $\bigcap_{k \in \mathbb{N}} G_k = \emptyset$. In particular,

$$\mathbb{N} = \bigcup_{k \in \mathbb{N}} G_k \setminus G_{k+1} \qquad \text{(disjoint union).}$$

Now, we construct for each $n \in \mathbb{N}$, an orthonormal basis $(\eta_k^n)_{k \in \mathbb{N}}$ for H_n as follows: let k be determined by $n \in G_k \setminus G_{k+1}$, and let $\{\eta_j^n \mid j = 1, ..., k\}$ be constructed from $\{\xi_j^n \mid j = 1, ..., k\}$ by the Gram–Schmidt orthonormalization process; then supply $\{\eta_j^n \mid j = 1, ..., k\}$ in any way to an orthonormal basis of H_n.

We now claim that

$$\lim_{n \to \omega} \|\eta_k^n - \xi_k^n\| = 0, \qquad k \in \mathbb{N}.$$

Namely, given k, then for each $n \in G_k$ one has that $\{\eta_j^n \mid j = 1, ..., k\}$ is constructed by the Gram–Schmidt process on $\{\xi_j^n \mid j = 1, ..., k\}$ and hence in particular

$$\eta_{j+1}^n \perp \operatorname{span}\{\xi_1^n, ..., \xi_j^n\}, \qquad j = 1, ..., k.$$

Choose $G \in \omega$ such that

$$|\langle \xi_i^n, \xi_j^n \rangle - \delta_{i,j}|$$

is small (with respect to some given $\varepsilon > 0$) for all $i, j \in \{1, ..., k\}$ and $n \in G$. Let $G' = G_k \cap G \in \omega$. Then for each $n \in G'$, it follows by continuity of the Gram–Schmidt process, and the fact that it does not change orthonormal sets, that $\|\xi_j^n - \eta_j^n\|$ is small (with respect to ε) for every $j \in \{1, ..., k\}$. This proves the claim.

Finally, let $u_n \colon K \to H_n$ be given by

$$u_n\left(\sum_i \lambda_i \xi_i\right) = \sum_i \lambda_i \eta_i^n, \qquad (\lambda_i) \in \ell^2(\mathbb{N}, \mathbb{C})$$

for all $n \in \mathbb{N}$. Clearly each u_n is unitary, and also for all $n \in \mathbb{N}$, and finite sums $\xi = \sum_i \lambda_i \xi_i \in K$, we have

$$\begin{aligned}\left\| u_n(\xi) - \sum_i \lambda_i \xi_i^n \right\| &= \left\| \sum_i \lambda_i (\eta_i^n - \xi_i^n) \right\| \\ &\leqslant \sum_i |\lambda_i| \cdot \|\eta_i^n - \xi_i^n\| \to 0 \qquad \text{as} \quad n \to \infty,\end{aligned}$$

so for such ξ, we get $\xi = (u_n \xi)_\omega$. Now for $\xi \in K$ arbitrary and $k \in \mathbb{N}$, choose $m \in \mathbb{N}$ and $\xi^{(k)} \in \operatorname{span}\{\xi_1, ..., \xi_m\}$ with $\|\xi - \xi^{(k)}\| < 1/k$. Let $\eta = (u_n \xi)_\omega$. Then as $\xi^{(k)} = (u_n \xi^{(k)})_\omega$, we have

$$\begin{aligned}\|\xi - \eta\| &\leqslant \|\xi - \xi^{(k)}\| + \|\xi^{(k)} - \eta\| \\ &\leqslant \frac{1}{k} + \lim_{n \to \omega} \|u_n \xi^{(k)} - u_n \xi\| \leqslant \frac{2}{k}, \qquad k \in \mathbb{N}.\end{aligned}$$

Hence $\xi = \eta$. ∎

We shall need to extend the notions of lim inf and lim sup in $\mathrm{vN}(H)$ slightly beyond what was mentioned in the introduction. We first recall the more general notions from [9, Section 2]. For a net $(M_\alpha) \subseteq \mathrm{vN}(H)$,

- an operator $x \in \mathcal{B}(H)$ is in the unit ball of $\liminf M_\alpha \in \mathrm{vN}(H)$ if and only if

$$U \cap \mathrm{Ball}(M_\alpha) \neq \emptyset$$

holds eventually, for every so*-neighborhood U of x;

- $\limsup M_\alpha$ is the von Neumann algebra generated by the set of operators $x \in \mathcal{B}(H)$ with the property that

$$U \cap \mathrm{Ball}(M_\alpha) \neq \emptyset$$

holds frequently, for every wo-neighborhood U of x.

We then have the convergence criterion [9, 2.8] and the commutant theorem [9, 3.5] which were mentioned in Section 1 in the context of sequences. We now extend the notions of lim inf and lim sup to convergence of nets along filters on the index set, as follows: given a net $(M_\alpha)_{\alpha \in I} \subseteq \mathrm{vN}(H)$ and a filter Φ on I, we define $\liminf_{\alpha \to \Phi} M_\alpha$ and $\limsup_{\alpha \to \Phi} M_\alpha$ exactly as above, except that the words "eventually" and "frequently" are now to be understood with respect to Φ; that is, to say that the property

$$U \cap \mathrm{Ball}(M_\alpha) \neq \emptyset$$

holds eventually, is to say that

$$\{\alpha \in I \mid U \cap \mathrm{Ball}(M_\alpha) \neq \emptyset\} \in \Phi,$$

and that the same property holds frequently means

$$\{\alpha \in I \mid U \cap \mathrm{Ball}(M_\alpha) = \emptyset\} \notin \Phi.$$

Notice that if Φ is an ultrafilter, then "eventually" and "frequently" (with respect to Φ) mean the same thing. In this paper, we only consider the case $\Phi = \omega$, where we have:

5.2. Lemma. *Let $(M_n) \subseteq \mathrm{vN}(H)$ be a sequence, and $M \in \mathrm{vN}(H)$. Then we have*

(i) $\liminf_{n \to \omega} M_n = \{x \in \mathcal{B}(H) \mid \exists (x_n) \in \prod_n M_n\colon \quad \sup_{n \in \mathbb{N}} \|x_n\| < \infty$ *and* $x_n \xrightarrow{\text{so*}} x \ (n \to \omega)\}$.

(ii) $\limsup_{n\to\omega} M_n = \{x \in \mathscr{B}(H) \mid \exists (x_n) \in \prod_n M_n\colon \ \sup_{n\in\mathbb{N}} \|x_n\| < \infty$ *and* $x_n \xrightarrow{\text{wo}} x\ (n\to\omega)\}''$.

(iii) $\limsup_{n\to\omega} M_n = (\liminf_{n\to\omega} M'_n)'$.

(iv) $M = \lim_{n\to\omega} M_n \Leftrightarrow \liminf_{n\to\omega} M_n = M = \limsup_{n\to\omega} M_n$.

Note. In (iv), $M = \lim_{n\to\omega} M_n$ means, of course, that $\lim_{n\to\omega} \|\varphi|_{M_n}\| = \|\varphi|_M\|$ for every $\varphi \in \mathrm{vN}(H)_*$.

Proof. In both (i) and (ii), the inclusion $\supseteq$ is trivial from the definitions. To prove $\subseteq$ in (i), we use that $\mathrm{Ball}(\mathscr{B}(H))$ is second countable in so*-topology, so that, given $x \in \mathrm{Ball}(\liminf_{n\to\omega} M_n)$, we may choose a basis for the so*-neighborhoods of x (relative to $\mathrm{Ball}(\mathscr{B}(H))$) consisting of a decreasing sequence (U_n). Then by the definition of $\liminf_{n\to\omega} M_n$, we have

$$F_n = \{n \in \mathbb{N} \mid U_m \cap \mathrm{Ball}(M_n) \neq \emptyset\} \in \omega, \qquad m \in \mathbb{N},$$

and also (F_m) forms a decreasing sequence. For each $m \in \mathbb{N}$ and $n \in F_m \setminus F_{m+1}$, we choose $x_n \in U_m \cap \mathrm{Ball}(M_n)$, and for each $n \in \mathbb{N}\setminus F_1$, we put $x_n = 0$. Then for every $m \in \mathbb{N}$ and every $n \in F_m$, we have $x_n \in U_m$, and thus $x_n \xrightarrow{\text{so*}} x$ as $n \to \omega$. This proves the inclusion $\subseteq$ in (i), and as "eventually" and "frequently" means the same thing with respect to ω, the same argument shows $\subseteq$ in (ii) (in fact the inclusion holds at the level of generating sets, like in (i)).

Finally, (iii) and (iv) are straightforward generalizations of the analogous results [9, 2.8 and 3.5] for nets. ∎

We next consider the (abstract) ultraproduct of a family of von Neumann algebras along ω, with respect to a sequence of their traces. Specifically, let $(M_n) \subseteq \mathrm{vN}(H)$, where H is fixed as usual, and let τ_n be a normal faithful tracial state on M_n for each $n \in \mathbb{N}$. Put

$$\mathscr{I}_\omega = \left\{ (x_n) \in \prod_n M_n \mid \sup_n \|x_n\| < \infty,\ \lim_{n\to\omega} \tau_n(x_n^* x_n) = 0 \right\},$$

$$\mathscr{M}^\omega = \left\{ (x_n) \in \prod_n M_n \mid \sup_n \|x\| < \infty \right\}.$$

One easily checks that $\mathscr{I}_\omega$ is a closed ideal in the C^*-algebra $\mathscr{M}^\omega$. We then introduce the following notation for the quotient C^*-algebra:

$$(M_n, \tau_n)^\omega = \mathscr{M}^\omega / \mathscr{I}_\omega,$$

and for any $(x_n) \in \mathscr{M}^\omega$, we denote by $(x_n)_\omega$ its image in $(M_n, \tau_n)^\omega$.

Notice that if $M_n = M \in \mathrm{vN}(H)$ and $\tau_n = \tau_m$ for each m and n in $\mathbb{N}$, then $(M_n, \tau_n)^\omega = M^\omega$ in the usual notation. Hence the following is an extension of the well known result for single algebras (see [5, Section 4] and [13, Chap. II, Theorems 6.2 and 7.1]) and follows by the same argument (which we do not repeat here):

5.3. THEOREM. *With the above notation, $(M_n, \tau_n)^\omega$ is a W^*-algebra, on which we have a normal faithful tracial state $(\tau_n)_\omega$, defined by*

$$(\tau_n)_\omega((x_n)_\omega) = \lim_{n\to\omega} \tau_n(x_n), \qquad (x_n)_\omega \in (M_n, \tau_n)^\omega.$$

We further introduce the following notations. Let $\tilde{H} = (L^2(M_n, \tau_n))_\omega$, $\tilde{M} = (M_n, \tau_n)^\omega$ and $\tilde{\tau} = (\tau_n)_\omega$. Define $w_2\colon \tilde{M}\xi_{\tilde{\tau}} \to \tilde{H}$ by

$$w_2(x_n)_\omega\, \xi_{\tilde{\tau}} = (x_n \xi_{\tau_n})_\omega, \qquad (x_n)_\omega \in \tilde{M}.$$

Then

$$\begin{aligned}\|w_2(x_n)_\omega\, \xi_{\tilde{\tau}}\| &= \lim_{n\to\omega} \|x_n\xi_{\tau_n}\| = \lim_{n\to\omega} \tau_n(x_n^* x_n)\\ &= \tilde{\tau}((x_n^* x_n)_\omega) = \|(x_n)_\omega\, \xi_{\tilde{\tau}}\|\end{aligned}$$

for all $(x_n)_\omega \in \tilde{M}$. Hence w_2 may be extended to an isometry of $L^2(\tilde{M}, \tilde{\tau})$ into $\tilde{H}$. Let J_{τ_n} and $J_{\tilde{\tau}}$ be the involutions on $L^2(M_n, \tau_n)$ and $L^2(\tilde{M}, \tilde{\tau})$, respectively, given by:

$$\begin{aligned} J_{\tau_n} x\xi_{\tau_n} &= x^*\xi_{\tau_n}, \qquad x \in M_n, \quad n \in \mathbb{N};\\ J_{\tilde{\tau}} x\xi_{\tilde{\tau}} &= x^*\xi_{\tilde{\tau}}, \qquad x \in \tilde{M}.\end{aligned}$$

We then define the antilinear isometry $(J_{\tau_n})_\omega$ on $\tilde{H}$ by

$$(J_{\tau_n})_\omega\, (\xi_n)_\omega = (J_{\tau_n}\xi_n)_\omega, \qquad (\xi_n)_\omega \in \tilde{H}.$$

5.4. LEMMA. *With the above notations, $w_2 J_{\tilde{\tau}} = (J_{\tau_n})_\omega\, w_2$.*

Proof. For $x \in (x_n)_\omega \in \tilde{M}$,

$$w_2 J_{\tilde{\tau}} x\xi_{\tilde{\tau}} = w_2(x_n^*)_\omega\, \xi_{\tilde{\tau}} = (x_n^*\xi_{\tau_n})_\omega = (J_{\tau_n})_\omega\, (x_n\xi_{\tau_n})_\omega = (J_{\tau_n})_\omega\, w_2 x\xi_{\tilde{\tau}}. \quad \blacksquare$$

5.5. LEMMA. *Let $\tilde{M} = (M_n, \tau_n)^\omega$ be as above and assume further that each M_n acts standardly on H. Let $M \in \mathrm{vN}^{\mathrm{st}}(H)$. Assume that we are given a normal unital *-monomorphism $i\colon M \to \tilde{M}$. Then there exist unitaries $(u_n) \subseteq \mathcal{U}(H)$ and a strictly increasing sequence $(n_k) \subseteq \mathbb{N}$ such that $u_{n_k} M_{n_k} u_{n_k}^* \to M$.*

Proof. Let $\tau_0 = \tilde{\tau} \circ i$. Then τ_0 is a tracial state on M; we let $E\colon \tilde{M} \to i(M)$ denote the trace preserving normal faithful conditional expectation. As M

and all M_n act standardly on H, we identify $L^2(M, \tau_0)$ and all $L^2(M_n, \tau_n)$ with H. Then we may extend i: $M \to \widetilde{M}$ to an isometry w_1: $H \to L^2(\widetilde{M}, \tilde{\tau})$ by the equation

$$w_1(x\xi_{\tau_0}) = i(x)\,\xi_{\tilde{\tau}}, \qquad x \in M,$$

because the definition of τ_0 ensures that this defines w_1 as an isometry on the dense subspace $M\xi_{\tau_0}$ of H. Let w_2 be as introduced above, and let $w = w_2 w_1$: $H \to \widetilde{H}$; also, let $K = wH$. By Lemma 5.1, we have unitaries v_n: $K \to L^2(M_n, \tau_n) = H$ satisfying

$$(v_n \xi)_\omega = \xi, \qquad \xi \in K.$$

Now let $(x_n) \in \prod_n M_n$ with $\sup \|x_n\| < \infty$. As ω is an ultrafilter and the unit ball of $\mathscr{B}(H)$ is wo-compact, we may define

$$x = \operatorname{wo-}\lim_{n \to \omega} v_n^* x_n v_n \in \mathscr{B}(K)$$

Let $u_n = w^* v_n^* \in \mathscr{U}(H)$, $n \in \mathbb{N}$. Then

$$w^* x w = \operatorname{wo-}\lim_{n \to \omega} u_n^* x_n u_n \in \limsup_{n \to \infty} u_n M_n u_n^*.$$

By Lemma 5.2(ii), it follows that elements of the form w^*xw (with x arising from (x_n) as above) generate $\limsup_{n \to \omega} u_n M_n u_n^*$. Hence, to prove $\limsup_{n \to \omega} u_n M_n u_n^* \subseteq M$, it suffices to prove:

$$w^* x w \in M.$$

To do so, let e: $L^2(\widetilde{M}, \tilde{\tau}) \to w_1 H$ be the projection given by

$$e x \xi_{\tilde{\tau}} = E(x)\,\xi_{\tilde{\tau}}, \qquad x \in \widetilde{M}.$$

Let $y = (x_n)_\omega \in \widetilde{M}$. Then for any norm-bounded sequence $(z_n) \in \prod_n M_n$,

$$w_2 y w_2^* (z_n \xi_{\tau_n})_\omega = w_2 y (z_n)_\omega\, \xi_{\tilde{\tau}} = (x_n z_n \xi_{\tau_n})_\omega,$$

so

$$w_2 y w_2^* (\xi_n)_\omega = (x_n \xi_n)_\omega, \qquad (\xi_n)_\omega \in K.$$

It follows that for $\xi, \eta \in K$,

$$\langle w_2 y w_2^* \xi, \eta \rangle = \lim_{n \to \omega} \langle x_n v_n \xi, v_n \eta \rangle = \lim_{n \to \omega} \langle v_n^* x_n v_n \xi, \eta \rangle = \langle x\xi, \eta \rangle.$$

As for every $a \in M$,

$$eyi(a)\,\xi_{\tau_0} = E(yi(a))\,\xi_{\tau_0} = E(y)\,i(a)\,\xi_{\tau_0},$$

and hence

$$E(y)\,\zeta = ey\zeta, \qquad \zeta \in W_1 H,$$

we now get, for all ξ, $\eta \in H$:

$$\begin{aligned}\langle xw\xi, w\eta\rangle &= \langle w_2 y w_2^* w_2 w_1 \xi, w_2 w_1 \eta\rangle \\ &= \langle y w_1 \xi, w_1 \eta\rangle \\ &= \langle e y w_1 \xi, w_1 \eta\rangle \\ &= \langle E(y)\, w_1 \xi, w_1 \eta\rangle \\ &= \langle w_1^* E(y)\, w_1 \xi, \eta\rangle,\end{aligned}$$

so

$$w_1^* E(y)\, w_1 = w^* x w.$$

But $E(y) \in i(M)$ and $w_1^* i(M)\, w_1 = M$, so this proves the claim, and hence, as we explained, we have $\limsup_{n\to\omega} u_n M_n u_n^* \subseteq M$.

The opposite inclusion follows using Lemma 5.4 and the above argument. Namely, as the involution $J_{\tilde{\tau}|_{i(M)}}$ on $w_1 H$ associated with the trace $\tilde{\tau}|_{i(M)}$ is given by $J_{\tilde{\tau}|_{i(M)}} = J_{\tilde{\tau}|_{w_1 H}}$, Lemma 5.4 gives

$$(J_{\tau_n})_\omega\, w\xi = w_2 J_{\tilde{\tau}} w_1 \xi = w_2 J_{\tilde{\tau}|_{i(M)}} w_1 \xi = w J_{\tau_0} \xi, \qquad \xi \in H,$$

and thus for all $\xi \in H$:

$$\begin{aligned}0 = \|wJ_{\tau_0}\xi - (J_{\tau_n})_\omega\, w\xi\| &= \lim_{n\to\omega} \|v_n w J_{\tau_0}\xi - J_{\tau_n} v_n w\xi\| \\ &= \lim_{n\to\omega} \|u_n^* J_{\tau_0}\xi - J_{\tau_n} u_n^*\xi\|,\end{aligned}$$

so we have

$$\mathrm{so}^* - \lim_{n\to\omega} (u_n J_{\tau_0} - J_{\tau_n} u_n^*) = 0.$$

Now, let $(x_n) \in \prod_n M'_n$ with $\sup \|x_n\| < \infty$. Define

$$x = \mathrm{wo} - \lim_{n\to\omega} v_n^* J_{\tau_n} x_n J_{\tau_n} v_n.$$

As $(J_{\tau_n} x_n J_{\tau_n}) \in \prod_n M_n$, we get from the previous paragraph that

$$w^* x w = \underset{n \to \omega}{\text{wo} - \lim} u_n J_{\tau_n} x_n J_{\tau_n} u_n^* \in M.$$

From the above,

$$\begin{aligned} J_{\tau_0} w^* x w J_{\tau_0} &= \underset{n \to \omega}{\text{wo} - \lim} J_{\tau_0} u_n J_{\tau_n} x_n J_{\tau_n} u_n^* J_{\tau_0} \\ &= \underset{n \to \omega}{\text{wo} - \lim} u_n J_{\tau_n}^2 x_n J_{\tau_n}^2 u_n^*, \end{aligned}$$

so

$$\underset{n \to \omega}{\text{wo} - \lim} u_n x_n u_n^* = J_{\tau_0} w^* x w J_{\tau_0} \in M'.$$

This proves $\limsup_{n \to \omega} u_n M_n' u_n^* \subseteq M'$. By Lemma 5.2 (iii)–(iv) and the previous paragraph, we may conclude $u_n M_n u_n^* \to M$ as $n \to \omega$. Since $\mathrm{vN}(H)$ is a separable metric space, we may then choose a subsequence as claimed. ∎

To prove the main theorem, in particular the equivalence mentioned at the beginning of the section, we still need two general lemmas concerning the possibility of extending normal faithful states to $\mathscr{B}(H)$.

5.6. LEMMA. *Let $M \in \mathrm{vN}(H)$ and let φ be a normal faithful state on M. Then there is a normal faithful state $\bar{\varphi}$ on $\mathscr{B}(H)$ such that $\bar{\varphi}|_M = \varphi$.*

Proof. As φ is positive and normal, there is a sequence $(\xi_n) \subseteq H$ such that

$$\varphi(x) = \sum_{n=1}^{\infty} \langle x \xi_n, \xi_n \rangle, \qquad x \in M.$$

Denote by φ_0 the normal state on $\mathscr{B}(H)$ given by this equation for all $x \in \mathscr{B}(H)$. Then φ_0 is normal, but possibly not faithful. Let $(u_n)_{n \in \mathbb{N}}$ be a so-dense countable subgroup of $\mathscr{U}(M')$, and put

$$\bar{\varphi}(x) = \sum_{n=1}^{\infty} 2^{-n} \varphi_0(u_n^* x u_n), \qquad x \in \mathscr{B}(H).$$

Then $\bar{\varphi}$ is normal and $\bar{\varphi}|_M = \varphi$. Also

$$\mathrm{s}(\bar{\varphi}) = \bigvee_{n=1}^{\infty} \mathrm{s}(u_n \varphi_0 u_n^*) = \bigvee_{n=1}^{\infty} u_n \mathrm{s}(\varphi_0)\, u_n^*,$$

where $\mathrm{s}(\cdot)$ denotes support projection. Hence

$$u_n \mathrm{s}(\bar{\varphi})\, u_n^* = \mathrm{s}(\bar{\varphi}), \qquad n \in \mathbb{N},$$

so that $\mathrm{s}(\bar{\varphi}) \in M'' = M$. But then

$$1 = \bar{\varphi}(\mathrm{s}(\bar{\varphi})) = \varphi(\mathrm{s}(\bar{\varphi})),$$

proving $\mathrm{s}(\bar{\varphi}) = 1$, as φ is faithful. ∎

The second lemma regards the simultaneous extension of dual states on crossed products; cf. [7] for the definition of dual states (or, more generally, weights).

5.7. Lemma. *Let $M \in \mathrm{vN}(H)$, let φ be a normal faithful state on $\mathscr{B}(H)$, and let $\varphi_0 = \varphi|_M$. Let $\bar{H} = H \otimes \ell^2(\mathbb{Z})$. Then there is a normal faithful state ψ on $\mathscr{B}(\ell^2(\mathbb{Z}))$ such that for any $\alpha \in \mathrm{Aut}(M)$ with $\varphi \circ \alpha = \varphi_0$, the dual state $\hat{\varphi}_0$ of φ_0 equals $\varphi \otimes \psi|_{M \rtimes_\alpha \mathbb{Z}}$.*

Proof. Let $(\varepsilon_n)_{n \in \mathbb{Z}}$ be an orthonormal basis of $\ell^2(\mathbb{Z})$, let $c_n = 2^{-|n|}/3$ $(n \in \mathbb{Z})$, and let

$$\psi(x) = \sum_{n \in \mathbb{Z}} c_n \langle x\varepsilon_n, \varepsilon_n \rangle, \qquad x \in \mathscr{B}(\ell^2(\mathbb{Z})).$$

Then ψ is a normal faithful state on $\mathscr{B}(\ell^2(\mathbb{Z}))$. Now let $\alpha \in \mathrm{Aut}(M)$ with $\varphi \circ \alpha = \varphi_0$. In matrix notation, $M \rtimes_\alpha \mathbb{Z}$ is the so-closure of finite sums of form

$$y = \sum_k \pi_\alpha(x_k)\, u^k,$$

where $x_k \in M$, $\pi_\alpha(x)_{ij} = \delta_{i,j} \alpha^{-i}(x)$, and $(u^k)_{ij} = \delta_{i,j+k} 1$; here, $\delta_{i,j}$ denotes Kronecker's delta. In this notation,

$$\hat{\varphi}_0(y) = \varphi(x_0),$$

so it suffices to prove

$$(\phi \otimes \psi)(\pi_\alpha(x)\, u^k) = \delta_{0,k} \varphi(x), \qquad x \in M, \quad k \in \mathbb{Z}.$$

Let $x \in M$, $k \in \mathbb{Z}$ and $y = \pi_\alpha(x)\, u^k$. Then

$$y_{ij} = \sum_{\ell \in \mathbb{Z}} \pi_\alpha(x)_{i\ell}\, u^k_{\ell j} = \sum_{\ell \in \mathbb{Z}} \delta_{i,l} \alpha^{-1}(x)\, \delta_{\ell, j+k} = \alpha^{-1}(x)\, \delta_{i, j+k}$$

so

$$y = \sum_{i \in \mathbb{Z}} \alpha^{-i}(x) \otimes e_{i,\, i-k},$$

where $(e_{i,\, j}) \subseteq \mathscr{B}(\ell^2(\mathbb{Z}))$ is the usual matrix unit with respect to the basis (ε_n). Hence

$$(\varphi \otimes \psi)(y) = \sum_{i \in \mathbb{Z}} \varphi(\alpha^{-i}(x))\, \psi(e_{i,\, i-k}) = \delta_{k,\, 0} \sum_{i \in \mathbb{Z}} c_i \varphi_0(x) = \delta_{k,\, 0} \varphi(x).$$

Q.E.D

By an *embedding* of an algebra into another, we understand a normal, unital, injective *-homomorphism.

We are now ready to prove:

5.8. Theorem. *If $N \in \mathscr{F}_{\mathrm{II}_1}$, the following statements are equivalent*:

(i) *N is in the closure of $\mathscr{F}_{\mathrm{inj}}$*

(ii) *There is an embedding $i\colon N \to R^\omega$.*

Proof. (i) $\Rightarrow$ (ii) Let τ be the tracial state on N, and extend τ to a normal faithful state φ on $\mathscr{B}(H)$, using Lemma 5.6. By (i), we may take $(N_n) \subseteq \mathscr{F}_{\mathrm{inj}}$ with $N_n \to N$. Let $\varphi_n = \varphi|_{N,\, n}$, $n \in \mathbb{N}$. Fix $T \in \mathbb{R} \backslash \{0\}$, and put (for all $n \in \mathbb{N}$):

$$\alpha_n = \sigma_T^{\varphi_n}, \qquad \tilde{N}_n = N_n \rtimes_{\alpha_n} \mathbb{Z}, \qquad \tilde{N} = N \rtimes_{\mathrm{id}} \mathbb{Z}.$$

These act on $\bar{H} = H \otimes \ell^2(\mathbb{Z})$, with common generating unitary $u = 1 \otimes \lambda_1$, where λ_1 is the left shift. Extend φ to $\bar{\varphi} = \varphi \otimes \psi$, where ψ is as in Lemma 5.7, so that

$$\bar{\varphi}|_{\tilde{N}} = \hat{\tau}, \qquad \bar{\varphi}|_{\tilde{N}_n} = \hat{\varphi}_n \qquad (n \in \mathbb{N}),$$

where $\hat{\tau}$, $\hat{\varphi}_n$ are the dual states. Take $h_0 \in \mathscr{B}(\ell^2(\mathbb{Z}))$ with $0 < h_0 \leqslant 2\pi$ and $h_0^{\mathrm{i}T} = \lambda_1$, hence with $h = h_0 \otimes 1 \in \mathscr{B}(\bar{H})$, one has $h^{\mathrm{i}T} = u$. Let

$$\chi(x) = \frac{\bar{\varphi}(h^{-1}x)}{\bar{\varphi}(h^{-1})}, \qquad x \in \mathscr{B}(\bar{H}).$$

Then χ is a normal faithful state on $\mathscr{B}(\bar{H})$. Defining

$$\psi_0 = \chi|_{\tilde{N}} \qquad \text{and} \qquad \psi_n = \chi|_{\tilde{N}_n} \qquad (n \in \mathbb{N}),$$

we get normal faithful states satisfying

$$\psi_n(x) = \frac{\hat{\varphi}_n(h^{-1}x)}{\hat{\varphi}(h^{-1})}, \qquad x \in N_n, \quad n \in \mathbb{N},$$

because $h \in \{u\}'' \subseteq \bigcap_n \tilde{N}_n$. It follows from the general theory of dual weights [7, 3.2] that

$$\sigma_T^{\psi_n} = \mathrm{ad}(u^*)\, \sigma_T^{\hat{\varphi}_n} = \mathrm{ad}(u^*)\, \mathrm{ad}(u)|_{N_n} = \mathrm{id}_{N_n}, \qquad n \in \mathbb{N},$$

and as ψ_0 is clearly a trace, also $\sigma_T^{\psi_0} = \mathrm{id}_N$. Now, let $Q_n = (\tilde{N}_n)_{\psi_n}$ be the centralizer of ψ_n for all $n \in \mathbb{N}$. Then each Q_n is a finite von Neumann algebra, since $\tau_n = \psi_n|_{Q_n}$ is a tracial state. Define

$$E_n(x) = \frac{1}{T}\int_0^T \sigma_t^{\psi_n}(x)\, dt, \qquad x \in \tilde{N}_n, \quad n \in \mathbb{N}.$$

Then each E_n is a normal faithful conditional expectation of $\tilde{N}_n$ onto Q_n. As $N_n \to N$, it follows from the proof of [9, 6.19] (with $\mathbb{Z}$ in the place of $\mathbb{R}$) that $\tilde{N}_n \to \tilde{N}$. Hence, by [9, 6.13], if $(x_n) \in \prod_n \tilde{N}_n$ and $x_n \xrightarrow{\text{so*}} x$, we have

$$\sigma_t^{\psi_n}(x_n) = \sigma_t^{\chi|\tilde{N}_n}(x_n) \xrightarrow{\text{so*}} \sigma_t^{\chi|\tilde{N}}(x) = x,$$

uniformly in t on compact subsets of $\mathbb{R}$. From this, it follows that

$$E_n(x_n) \xrightarrow{\text{so*}} \frac{1}{T}\int_0^T x\, dt = x \qquad \text{as} \quad n \to \infty.$$

So if $x \in \tilde{N} = \lim_{n\to\infty} \tilde{N}_n$, we may take $(x_n) \in \prod_n \tilde{N}_n$ such that $x_n \xrightarrow{\text{so*}} x$, and define $i_0\colon \tilde{N} \to (Q_n, \tau_n)^\omega$ by

$$i_0(x) = (E_n(x_n))_\omega.$$

It is apparent that $(\tau_n)_\omega \circ i_0 = (\psi_n)_\omega$, so i_0 is normal; also, i_0 is easily checked to be a unital *-monomorphism. As N can be embedded trivially in $\tilde{N} \cong N \otimes \ell^\infty(\mathbb{Z})$, we have an embedding of N into $(Q_n, \tau_n)^\omega$. Notice that each $\tilde{N}_n$ is injective because each N_n is so, and hence each Q_n is injective as it is expected from $\tilde{N}_n$ by E_n. So if $Q = \bigotimes_n (Q_n, \tau_n)$ is the infinite tensor power, then Q is finite and injective, and hence $Q \rtimes \mathbb{S}_\infty$ (where the crossed product is by the permutation action, as in the proof of Theorem 2.6) is an injective factor of type II_1. Hence for each n, there is an embedding i_n of Q_n into R such that $\tau_R \circ i_n = \tau_n$, where τ_R is the tracial. state on R; but then $(Q_n, \tau_n)^\omega$ embeds in R^ω via $(i_n)_\omega$. This together with the embedding of N into $(Q_n, \tau_n)^\omega$ shows (ii).

(ii) $\Rightarrow$ (i) This is clear from Lemma 5.5 in the special case where $N \in \mathscr{F}^{\mathrm{st}}_{\mathrm{II}_1}$, i.e. when $\dim_H(N) = 1$, where $\dim_H(N)$ denotes the coupling constant (cf. e.g. [4, Section III.6]).

Assume next $c = \dim_H(N) < \infty$. Choose $M \in \mathscr{F}_{\mathrm{II}_1} \cap \mathscr{F}_{\mathrm{inf}}$ such that $\dim_H(M) = 1/c$. Then $\dim_H(N \otimes M) = 1$ so $N \otimes M$ is a type II_1-factor acting standardly on $H \otimes H$. Also, as N embeds in R^ω, so does $N \otimes M$ (since $(R \otimes M)^\omega \cong (R \otimes R)^\omega \cong R^\omega$). By Lemma 2.4, N can be approximated by factors acting on H which are spatially isomorphic to $N \otimes M$, hence N is in the closure of $\mathscr{F}^{\mathrm{st}}_{\mathrm{II}_1}$ and may be embedded in R^ω. By the first part of the proof, each of the standard approximants of N are in the closure of $\mathscr{F}_{\mathrm{inj}}$, hence so is N.

Finally, if $\dim_H(N) = \infty$, i.e. $N' \in \mathscr{F}_{\mathrm{II}_\infty}$, we choose an ascending sequence $(p_n) \subseteq N'$ of finite but infinite dimensional projections, such that $p_n \to 1$. Then $Np_n \cong N$ and $\dim_{p_n H}(Np_n) < \infty$, and Np_n clearly embeds in R^ω for each n. Hence by the second part of the proof, each Np_n may be approximated by injective factors acting on $p_n H$. On the other hand, with (v_n) chosen as in Lemma 3.2 (i.e. $v_n^* v_n = 1$, $v_n v^* = p_n$ $(n \in \mathbb{N})$ and $v_n \xrightarrow{\mathrm{so}} 1$), one has that $v_n^* N p_n v_n = v_n^* N v_n \to N$, and by the above, each $v_n^* N p_n v_n$ is in the closure of $\mathscr{F}_{\mathrm{inj}}$. ∎

The previous theorem yields in particular the equivalence of the statements mentioned at the beginning of the section:

5.9. Corollary. *The following statements are equivalent*:

(C) $\mathscr{F}_{\mathrm{inj}}$ *is dense in* $\mathrm{vN}(H)$.

(E) *For all* $N \in \mathscr{F}_{\mathrm{II}_1}$, *there is an embedding* $i \colon N \to R^\omega$.

Proof. (C) $\Rightarrow$ (E): Immediate from Theorem 5.8.

(E) $\Rightarrow$ (C): As the factors of type II_1 are dense in $\mathrm{vN}(H)$ by Corollary 3.6, this follows also from Theorem 5.8. Q.E.D

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